

# The Stokes Phenomenon and Quantum Tunneling for de Sitter Radiation in Nonstationary Coordinates

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**Sang Pyo Kim**

*Department of Physics, Kunsan National University, Kunsan 573-701, Korea*

*Asia Pacific Center for Theoretical Physics, Pohang 790-784, Korea*

*E-mail: sangkim@kunsan.ac.kr*

**ABSTRACT:** We study quantum tunneling for the de Sitter radiation in the planar coordinates and global coordinates, which are nonstationary coordinates and describe the expanding geometry. Using the phase-integral approximation for the Hamilton-Jacobi action in the complex plane of time, we obtain the particle-production rate in both coordinates and derive the additional sinusoidal factor depending on the dimensionality of spacetime and the quantum number for spherical harmonics in the global coordinates. This approach resolves the factor of two problem in the tunneling method.

**KEYWORDS:** Black Holes, Black Holes in String Theory, Field Theories in Higher Dimensions, Nonperturbative Effects.

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## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. Schwinger Mechanism in Time-Dependent Gauge</b>	<b>2</b>
<b>3. de Sitter Spacetimes</b>	<b>4</b>
<b>4. Conclusion</b>	<b>6</b>

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## 1. Introduction

Black hole physics has been a key issue in gravity since Hawking's discovery of black hole radiation [1]. Black holes radiate thermal spectra with the temperature determined by the surface gravity at the event horizon. One method to get the Hawking radiation is the Bogoliubov transformation between the in-falling state and the out-going state scattered by the event horizon [1]. Recently Parikh and Wilczek have suggested an intuitive interpretation of the Hawking radiation as quantum tunneling through the event horizon of particles produced from vacuum fluctuations near the horizon [2]. The tunneling interpretation has revitalized the study of Hawking radiation in various black holes. Other phenomena for particle production are the Schwinger mechanism [3] and the expanding geometry [4]. The Schwinger mechanism can be understood as quantum tunneling process of virtual pairs from the Dirac sea due to the electric field.

The tunneling method has been mostly applied to static or stationary black holes, where the event horizon provides a causal boundary for tunneling of particles from the interior to the exterior. Then a question may be raised whether the tunneling process is unique to the static or stationary black holes. The de Sitter spacetime, describing the geometry of an expanding spacetime, has a few coordinate systems, one of which is the static coordinates that exhibit the cosmological event horizon with the Gibbons-Hawking temperature [5]. The tunneling method has been applied to the static coordinates of de Sitter radiation [6, 7, 8, 9, 10]. In the static coordinates of the de Sitter spacetime in four dimensions [11, 12]

$$ds^2 = -\left(1 - \frac{r^2}{r_H^2}\right)dt^2 + \frac{dr^2}{1 - \frac{r^2}{r_H^2}} + r^2 d\Omega_2^2, \quad (1.1)$$

the cosmological event horizon is located at  $r_{dS} = r_H$  and has the Gibbons-Hawking temperature  $T_{dS} = 1/(2\pi r_H)$ . Volovik used the fluid metric for de Sitter spacetime, which is still a stationary coordinates system, and derived the de Sitter radiation via quantum tunneling [13].

In this paper we study the particle emission via tunneling in nonstationary coordinates for de Sitter spacetimes. Since Parker has shown that expanding spacetimes also create particles [4], the particle-production rate in de Sitter spacetimes has been known for many years [14, 15]. The WKB approximation in complex time has been employed for the de Sitter radiation [16]. In the Schwinger mechanism charged particle pairs can also be produced by time-dependent electric fields or a constant electric field in time-dependent gauge. Hence it is worthy to investigate the de Sitter radiation via tunneling in the planar coordinates and the global coordinates, these being nonstationary coordinates. Other motivation is the derivation of the Friedmann equation for de Sitter spacetime from the first law of black hole thermodynamics [17].

In expanding spacetimes or time-dependent electric fields, the ingoing positive frequency solution of a quantum field splits into the outgoing positive and negative frequency solutions. The ratio of the outgoing negative solution to the ingoing positive solution determines the mean number of produced particles [4]. The mean number is approximately given by the imaginary part of the Hamilton-Jacobi action, which in turn is determined by a contour integral in the complex plane of time [18]. This approach based on the phase-integral approximation [19] generalizes the tunneling idea to time-dependent electric fields [18]. Other field theoretical methods are also attempted in time-dependent contexts [20, 21, 22].

We apply the phase-integral approximation to the planar and global coordinates of de Sitter spacetime and obtain the de Sitter radiation. The phase-integral approximation for quantum tunneling recovers not only the Boltzmann factor for the particle-production rate but also from the Stokes phenomenon [23] the sinusoidal factor that depends on the dimensionality of spacetime and the quantum number for spherical harmonics. One controversial issue in the tunneling method is the factor of two problem, according to which some coordinates give the temperature for radiation twice as big as the Hawking temperature [24, 25, 26, 27]. We show that the tunneling method in nonstationary coordinates properly yields the Hawking temperature and thus resolves the factor of two problem.

The organization of this paper is as follows. In section 2 we briefly review the Schwinger mechanism in the time-dependent gauge and formulate the pair-production rate via quantum tunneling in the complex time. In section 3 we derive the de Sitter radiation via quantum tunneling in the planar and global coordinates and explain the origin of sinusoidal factor that depends on dimensionality of spacetime and quantum number for spherical harmonics. Finally, we summarize the tunneling method in nonstationary spacetimes and discuss the problem of the factor of two in section 4.

## 2. Schwinger Mechanism in Time-Dependent Gauge

The quantum electrodynamics (QED) analog for static or stationary black holes is the Coulomb gauge, in which the field equation becomes the tunneling problem. The analogy between the Schwinger mechanism and the Hawking radiation has been studied in Refs. [28, 29, 30]. In QED the pair-production rate can be found in various methods, such as the phase-integral approximation [31, 32, 18] and the worldline instanton method [33, 34]. The

constant electric field has not only the Coulomb gauge but also the time-dependent gauge, in which the field equation becomes either tunneling under the barrier or transmission over the barrier.

We briefly review the phase-integral approximation for pair production in the time-dependent gauge [18], which is the QED analog for nonstationary spacetimes. The component of a charged scalar field in the gauge potential  $A_{\parallel}(t)$  satisfies the equation

$$\ddot{\phi}_{\mathbf{k}}(t) + Q_{\mathbf{k}}(t)\phi_{\mathbf{k}}(t) = 0, \quad (2.1)$$

where

$$Q_{\mathbf{k}}(t) = m^2 + \mathbf{k}_{\perp}^2 + (k_{\parallel} + qA_{\parallel}(t))^2. \quad (2.2)$$

Here  $m$  and  $q$  are the mass and charge of the particle, and  $\mathbf{k}_{\perp}$  and  $k_{\parallel}$  are the momentum component transverse and parallel to the electric field, respectively. The magnitude square of the ratio of the outgoing negative frequency solution to the ingoing positive solution is the mean number of produced pairs. In Ref. [18], the WKB instanton action for the Hamilton-Jacobi equation

$$2\text{Im}S_{\mathbf{k}} = i \oint \sqrt{Q_{\mathbf{k}}(t)} dt, \quad (2.3)$$

where the contour integral is taken outside of a loop enclosing two roots of  $Q_{\mathbf{k}}(t)$  in the complex plane of time, approximately gives the mean number of produced pairs

$$\bar{n}_{\mathbf{k}} = e^{-2\text{Im}S_{\mathbf{k}}}. \quad (2.4)$$

The action  $2\text{Im}S_{\mathbf{k}}$  takes the probability into account, which is equivalent to the detailed balance of the emission rate to the absorption rate [28, 21].

The instability of the vacuum due to pair production is characterized by the vacuum persistence, which is related to the mean number of produced pairs. The vacuum persistence for bosons is the probability for the in-vacuum to remain in the out-vacuum [35, 36]

$$|\langle \text{out} | \text{in} \rangle|^2 = e^{-VT \sum_{\mathbf{k}} \ln(1 + \bar{n}_{\mathbf{k}})}, \quad (2.5)$$

where  $V$  is the volume and  $T$  is the duration. For small pair production  $\bar{n}_{\mathbf{k}} \ll 1$ , the decay rate per unit volume and per unit time is approximately given by the total mean number of produced pairs:

$$\frac{\Gamma}{VT} = 1 - |\langle \text{out} | \text{in} \rangle|^2 \approx \sum_{\mathbf{k}} \bar{n}_{\mathbf{k}}. \quad (2.6)$$

In the phase-integral approximation for tunneling, it is understood that the emission rate is determined by Eq. (2.4).

### 3. de Sitter Spacetimes

The de Sitter spacetime has a few different coordinate systems. The  $(d+1)$ -dimensional de Sitter spacetime has the planar coordinates,<sup>1</sup>

$$ds^2 = -dt^2 + e^{2Ht} d\mathbf{x}_d^2, \quad (3.1)$$

where the units  $c = \hbar = 1$  are used, and also has the global coordinates

$$ds^2 = -dt^2 + \frac{1}{H^2} \cosh^2(Ht) d\Omega_d^2. \quad (3.2)$$

The field equation for a scalar field contains the  $d$ -dimensional Laplace operator and harmonics [37]

$$\nabla_d^2 u_\kappa(\mathbf{x}) = -\kappa^2 u_\kappa(\mathbf{x}), \quad (3.3)$$

where for the planar coordinates  $\nabla_d^2$  acts on the  $d$ -dimensional Euclidean space and has  $\kappa^2 = \mathbf{k}^2$  while for the global coordinates  $\nabla_d^2$  acts on  $S_d$  and has  $\kappa^2 = l(l+d-1)$ , ( $l = 0, 1, \dots$ ). Then the harmonic-components of the scalar field,  $\Phi(t, \mathbf{x}) = a^{-d/2} \sum_\kappa u_\kappa(\mathbf{x}) \phi_\kappa$ , satisfy

$$\ddot{\phi}_\kappa(t) + Q_\kappa(t) \phi_\kappa(t) = 0, \quad (3.4)$$

where

$$Q_\kappa(t) = m^2 + \frac{\kappa^2}{a^2} - \frac{d}{2} \left( \frac{d}{2} - 1 \right) \left( \frac{\dot{a}}{a} \right)^2 - \frac{d}{2} \frac{\ddot{a}}{a}. \quad (3.5)$$

Here  $a(t) = e^{Ht}$  for the planar coordinates and  $a(t) = \cosh(Ht)/H$  for global coordinates. We now apply the phase-integral approximation and find the solution of the form

$$\ddot{\phi}_\kappa(t) = e^{-iS_\kappa(t)}, \quad (3.6)$$

in terms of the Hamilton-Jacobi action

$$S_\kappa(t) = \int \sqrt{Q_\kappa(t)} dt. \quad (3.7)$$

The imaginary part of  $S_\kappa$  is responsible for the decay rate and gives the Boltzmann factor for the de Sitter radiation,

$$\Gamma_\kappa = |\phi_\kappa|^2 = e^{-2\text{Im}S_\kappa}. \quad (3.8)$$

First, in the planar coordinates, Eq. (3.7) takes the form

$$S_{\mathbf{k}}(t) = \int \sqrt{\gamma^2 + \mathbf{k}^2 e^{-2Ht}} dt, \quad \gamma = \sqrt{m^2 - \frac{(dH)^2}{4}}. \quad (3.9)$$

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<sup>1</sup>The planar coordinates are also called the oblique coordinates and the global coordinates are the horizontal coordinates [12].

Changing the variable  $\tau = e^{Ht}$  and integrating from a turning point  $\tau_- = -ik/\gamma$  to another  $\tau_+ = ik/\gamma$ , we obtain a pure imaginary action

$$S_{\mathbf{k}} = \frac{1}{H} \int_{-i\frac{k}{\gamma}}^{i\frac{k}{\gamma}} \frac{d\tau}{\tau^2} \sqrt{\mathbf{k}^2 + \gamma^2 \tau^2} = i\pi \frac{\gamma}{H}. \quad (3.10)$$

The analogy with the Schwinger mechanism suggests another way to find the instanton action. The production rate of particles may be given by a contour integral [18]

$$2\text{Im}S_{\mathbf{k}} = i \oint \sqrt{\gamma^2 + \mathbf{k}^2} e^{-2Ht} dt, \quad (3.11)$$

in the complex plane of time  $t$ . Expanding for large  $\tau$  and choosing a contour exterior to two branch cuts  $\tau_{\pm}$ , the contour integral

$$2\text{Im}S_{\mathbf{k}} = i \frac{\gamma}{H} \oint \sqrt{1 + \frac{\mathbf{k}^2}{(\gamma\tau)^2}} \frac{d\tau}{\tau}, \quad (3.12)$$

with the negative sign for residues, leads to the instanton action

$$2\text{Im}S_{\mathbf{k}} = 2\pi \frac{\gamma}{H}. \quad (3.13)$$

Second, in the global coordinates, the instanton action of the Hamilton-Jacobi equation is given by

$$S_l = \int \sqrt{\gamma^2 + \frac{(\lambda H)^2}{\cosh^2(Ht)}} dt, \quad \lambda^2 = l(l+d-1) + \frac{d}{2} \left( \frac{d}{2} - 1 \right). \quad (3.14)$$

There are four turning points,

$$\begin{aligned} e^{Ht_{(a)\pm}} &= -i(\lambda H/\gamma) \pm i\sqrt{(\lambda H/\gamma)^2 + 1}, \\ e^{Ht_{(b)\pm}} &= +i(\lambda H/\gamma) \pm i\sqrt{(\lambda H/\gamma)^2 + 1}. \end{aligned} \quad (3.15)$$

These are grouped into complex conjugate pairs  $\{t_{(a)+}, t_{(b)-}\}$  and  $\{t_{(a)-}, t_{(b)+}\}$ . Changing the variable to a conformal time,  $\sinh(Ht) = \tan(\tau)$ , the Hamilton-Jacobi action takes the form

$$S_l(t_{(a)}, t_{(b)}) = \frac{\gamma}{H} \sqrt{1 + \frac{(\lambda H)^2}{\gamma^2}} \int_{t_{(a)}}^{t_{(b)}} \sqrt{1 - \frac{\frac{(\lambda H)^2}{\gamma^2} \sin^2(\tau)}{1 + \frac{(\lambda H)^2}{\gamma^2} \cos(\tau)}} \frac{d\tau}{\cos(\tau)}. \quad (3.16)$$

The integral can be done [38]

$$S_l(t_{(a)}, t_{(b)}) = i\pi \frac{\gamma}{H} + \pi\lambda. \quad (3.17)$$

Each complex conjugate pair contributes  $2\text{Im}S_l = 2\pi\gamma/H$  and the crossing integral, for instance, from  $t_{(a)-}$  to  $t_{(b)-}$  yields the real part  $\text{Re}S_l = \pi\lambda$ . Therefore, the Stokes phenomenon, Eq. (7) of Ref. [23] (see also Ref. [19]), gives the particle-production rate

$$\begin{aligned} \bar{n}_l &\approx e^{-2\text{Im}S_l(t_{(a)+}, t_{(b)-})} + e^{-2\text{Im}S_l(t_{(a)-}, t_{(b)+})} \\ &\quad + 2\cos(2\text{Re}S_l(t_{(a)-}, t_{(b)-})) e^{-\text{Im}S_l(t_{(a)+}, t_{(b)-}) - \text{Im}S_l(t_{(a)-}, t_{(b)+})}. \end{aligned} \quad (3.18)$$

In the asymptotic limit of large action ( $S_l \gg 1$  and  $l \gg 1$ ), we approximate  $\lambda \approx l + (d-1)/2$  and obtain

$$\bar{n}_l \approx 4 \sin^2\left(\pi\left(l + \frac{d}{2}\right)\right) e^{-2\pi \frac{\gamma}{H}}, \quad (3.19)$$

which is the leading term of particle-production rate in the  $(d+1)$ -dimensional de Sitter spacetime

$$\bar{n}_l = \frac{\sin^2(\pi(l + \frac{d}{2}))}{\sinh^2(\frac{\pi\gamma}{H})}. \quad (3.20)$$

The sinusoidal factor is a consequence of the substructure of the Stokes phenomenon and explains the absence of particle production in odd dimensions [15, 39]. In the second approach, changing the variable  $u = \cosh(Ht)/H$  and expanding for large  $u$ , the contour integral leads to

$$2\text{Im}S_l = i \frac{\gamma}{H} \oint \frac{d\tau}{\tau} \frac{\sqrt{1 + \frac{\lambda^2}{(\gamma\tau)^2}}}{\sqrt{1 - \frac{1}{(H\tau)^2}}} = 2\pi \frac{\gamma}{H}. \quad (3.21)$$

In summary, we applied the phase-integral approximation to the planar coordinates and the global coordinates of de Sitter spacetime and obtained the Boltzmann factor for the mean number of produced particles

$$\bar{n}_\kappa = e^{-2\pi \frac{\gamma}{H}}. \quad (3.22)$$

The de Sitter radiation has the Gibbons-Hawking temperature  $T_{\text{dS}} = H/2\pi$ . In the global coordinates, in addition to the Boltzmann factor, there is a sinusoidal factor that depends on the spacetime dimensions and the quantum number for spherical harmonics. That particles are not produced in odd dimensional de Sitter spacetimes is a consequence of the Stokes phenomenon in the global coordinates.

## 4. Conclusion

In this paper we have extended the tunneling idea for particle production to nonstationary coordinates of de Sitter spacetimes. The tunneling method, which provides a physical intuition to the Hawking radiation of black holes, has been also applied to static coordinates of de Sitter spacetimes. It is thus interesting to investigate the tunneling method in nonstationary spacetimes. For this purpose we have used the analogy of particle production by expanding spacetimes with the Schwinger mechanism by electric fields.

In QED a constant electric field has the Coulomb gauge or the time-dependent gauge. The Coulomb gauge corresponds to a static black hole, while the time-dependent gauge corresponds to a nonstationary spacetime. Further, the pair-production rate for a given momentum  $\mathbf{k}$

$$\bar{n}_{\mathbf{k}} = e^{-2\text{Im}S_{\mathbf{k}}} \quad (4.1)$$

is given by the Hamilton-Jacobi action in the complex plane of space or time [18]

$$2\text{Im}S_{\mathbf{k}} = \mp i \oint \sqrt{Q_{\mathbf{k}}(z)} dz. \quad (4.2)$$

Here the upper sign is for the Coulomb gauge and  $Q_{\mathbf{k}}(z)$  is the kinematic momentum in the electric field while the lower sign is for the time-dependent gauge and  $Q_{\mathbf{k}}(z)$  is the kinematic energy. In this sense Eqs. (4.1) and (4.2) provide a unified tunneling method for the Coulomb gauge and the time-dependent gauge.

We have shown that the tunneling method formulated by Eqs. (4.1) and (4.2) also applies to the planar and global coordinates of de Sitter spacetimes. The tunneling method recovers the Boltzmann factor for de Sitter radiation as shown in section 3 and, to our surprise, it yields the sinusoidal factor depending on the dimensionality of de Sitter spacetime and the quantum number for spherical harmonics in the global coordinates. The absence of particle production in the global coordinates of odd dimensional de Sitter spacetimes is a consequence of the Stokes phenomenon.

One controversial issue in the tunneling method is the factor of two problem [24, 25, 26, 27], where the temperature is twice as big as the Hawking temperature. To resolve this problem, either the isotropic coordinates [40, 41] or Rindler coordinates [30] have been used. Indeed, in these coordinates the tunneling method with the upper sign of Eqs. (4.1) and (4.2) yields correctly the de Sitter radiation. However, tunneling method does not have such an ambiguity in the planar and global coordinates as shown in section 3.

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